

Unipotent Hecke algebras of $\mathrm{GL}_n(\mathbb{F}_q)$

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Abstract

This paper describes a family of Hecke algebras $\mathcal{H}_\mu = \mathrm{End}_G(\mathrm{Ind}_U^G(\psi_\mu))$, where U is the subgroup of unipotent upper-triangular matrices of $G = \mathrm{GL}_n(\mathbb{F}_q)$ and ψ_μ is a linear character of U . The main results combinatorially index a basis of \mathcal{H}_μ , provide a large commutative subalgebra of \mathcal{H}_μ , and after describing the combinatorics associated with the representation theory of \mathcal{H}_μ , generalize the RSK correspondence that is typically found in the representation theory of the symmetric group.

1 Introduction

Iwahori [Iw] and Iwahori-Matsumoto [IM] introduced the Iwahori-Hecke algebra as a first step in classifying the irreducible representations of finite Chevalley groups and reductive p -adic Lie groups. Subsequent work (e.g. [Cu1] [KL] [LV]) has established Hecke algebras as fundamental tools in the representation theory of Lie groups and Lie algebras, and advances on subfactors and quantum groups by Jones [Jo1], Jimbo [Ji], and Drinfeld [Dr] gave Hecke algebras a central role in knot theory [Jo2], statistical mechanics [Jo3], mathematical physics, and operator algebras. This paper considers a generalization of the classical Iwahori-Hecke algebra obtained by replacing the Borel subgroup B with the unipotent subgroup U .

A Hecke algebra $\mathcal{H} = \mathcal{H}(G, U, M)$ is the centralizer algebra

$$\mathcal{H} = \mathrm{End}_G(\mathrm{Ind}_U^G(M)),$$

where G is a finite group, U is a subgroup of G and M is a simple U -module. This paper addresses the cases where

$$G = \mathrm{GL}_n(\mathbb{F}_q), \quad U = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}, \quad \text{and} \quad \dim(M) = 1.$$

For each M there exist a partition $\mu = (\mu_1, \mu_2, \dots, \mu_\ell)$ and a linear character $\psi_\mu : U \rightarrow \mathbb{C}^*$ such that

$$\mathrm{End}_G(\mathrm{Ind}_U^G(M)) \cong \mathrm{End}_G(\mathrm{Ind}_U^G(\psi_\mu)) = \mathcal{H}_\mu.$$

MSC 2000: 20C08 (05E10)

Keywords: Hecke algebras, Gelfand-Graev representation, Yokonuma algebra, RSK-insertion

In studying these algebras, this paper combines generalizations of Iwahori-Hecke algebra techniques with combinatorial tools related to the representation theory of the symmetric group (e.g. partitions, tableaux, and the RSK correspondence).

The main results of this paper are

- (a) Let $\{T_v : v \in N_\mu\}$ be the standard double-coset basis for \mathcal{H}_μ . Then there is a bijection

$$N_\mu \longleftrightarrow M_\mu = \left\{ \begin{array}{l} \ell \times \ell \text{ matrices with monic polynomial} \\ \text{entries } a_{ij}(X) \in \mathbb{F}_q[X] \text{ such that} \\ a_{ij}(0) \neq 0 \text{ and both degree row sums} \\ \text{and degree column sums are equal to } \mu \end{array} \right\}$$

(See Section 3 for details).

- (b) Let the set $\hat{\mathcal{H}}_\mu$ index the irreducible \mathcal{H}_μ -modules \mathcal{H}_μ^λ and the set $\hat{\mathcal{H}}_\mu^\lambda$ index a basis of the irreducible module \mathcal{H}_μ^λ . There is a combinatorial bijection

$$N_\mu \longleftrightarrow \left\{ \begin{array}{l} \text{Pairs } (P, Q) \text{ such} \\ \text{that } P, Q \in \mathcal{H}_\mu^\lambda, \lambda \in \hat{\mathcal{H}}_\mu \end{array} \right\}$$

that generalizes the classical RSK correspondence and gives a combinatorial realization of the representation theoretic identity

$$\dim(\mathcal{H}_\mu) = \sum_{\lambda \in \hat{\mathcal{H}}_\mu} \dim(\mathcal{H}_\mu^\lambda)^2.$$

- (c) The algebra \mathcal{H}_μ has a large commutative subalgebra

$$\mathcal{L}_\mu \cong \mathcal{H}_{(\mu_1)} \otimes \mathcal{H}_{(\mu_2)} \otimes \cdots \otimes \mathcal{H}_{(\mu_\ell)},$$

whose presence suggests a weight space decomposition of \mathcal{H}_μ -modules.

Section 2 reviews some basic results used in this paper, including Hecke algebras, partitions and the classical RSK correspondence. Section 3 defines \mathcal{H}_μ and gives an explicit construction of the map $N_\mu \leftrightarrow M_\mu$. After using Zelevinsky's theorem [Ze, Theorem 12.1] to give a combinatorial description of the sets $\hat{\mathcal{H}}_\mu$ and $\hat{\mathcal{H}}_\mu^\lambda$, Section 4 proves the bijection in (b). Section 5 gives a proof of Zelevinsky's theorem. This paper concludes in Section 6 by providing the subalgebra $\mathcal{L}_\mu \subseteq \mathcal{H}_\mu$ and describing a corresponding weight space decomposition of \mathcal{H}_μ -modules.

Both the Yokonuma algebra $\mathcal{H}_{(1^n)}$ [Yo2] and the Hecke algebra $\mathcal{H}_{(n)}$ associated to the Gelfand-Graev representation of G [St] are examples of unipotent Hecke algebras. In [Yo2], Yokonuma gave a presentation of $\mathcal{H}_{(1^n)}$ that generalized the usual presentation of the classical Iwahori-Hecke algebra, and recently Jujumaya [Ju] constructed an alternate set of generators and relations. However, a presentation for arbitrary \mathcal{H}_μ is still unknown. Even the commutative algebra $\mathcal{H}_{(n)}$ [St, Yo1] does not yet have a “nice” set of multiplication relations.

The representation theory of the Gelfand-Graev Hecke algebra is closely related to Green polynomials [Cu2] and, in the GL_2 case, to Kloosterman sums [CS]. On the other hand, the representation theory of the Yokonuma algebra generalizes that of the classical Iwahori-Hecke algebra. In what promises to be a combinatorially rich area of study, analyzing the combinatorial representation theory of the \mathcal{H}_μ and their general type analogues should have an impact similar in scope to the applications of the classical Iwahori-Hecke algebra.

Acknowledgements. This paper will be a portion of my Ph.D. thesis. In developing these results, I have enjoyed the supportive environment of the University of Wisconsin-Madison mathematics department, the time supplied by several grants (VIGRE DMS-9819788, NSF DMS-0097977, and NSA MDA904-01-1-0032), and above all the patient help and insights of my advisor Arun Ram.

2 Preliminaries: Hecke algebras, some combinatorics of the symmetric group, and $\mathrm{GL}_n(\mathbb{F}_q)$

Hecke algebras. Let U be a subgroup of a finite group G . If M is an irreducible U -module, then the *Hecke algebra* $\mathcal{H} = \mathcal{H}(G, U, M)$ is

$$\mathcal{H} = \mathrm{End}_G(\mathrm{Ind}_U^G(M)) \cong e\mathbb{C}Ge,$$

where e is an idempotent of $\mathbb{C}U$ such that $M \cong \mathbb{C}Ue$ [CR, (3.19)]. If N is a set of double coset representatives for the cosets $U \backslash G / U$, then the set

$$\{T_v = eve : v \in N, eve \neq 0\} \quad (2.1)$$

is a basis for \mathcal{H} [CR, (11.30)].

Let $\hat{\mathcal{H}}$ be an indexing set for the irreducible \mathcal{H} -modules \mathcal{H}^λ . As a (G, \mathcal{H}) -bimodule,

$$\mathbb{C}Ge \cong \mathrm{Ind}_U^G(M) \cong \bigoplus_{\lambda \in \hat{\mathcal{H}}} G^\lambda \otimes \mathcal{H}^\lambda,$$

where the G^λ are the irreducible constituents of $\mathrm{Ind}_U^G(M)$ [GW, Thm 3.3.7]; it follows that

$$\dim(\mathcal{H}^\lambda) = \text{multiplicity of } G^\lambda \text{ in the } G\text{-module } \mathrm{Ind}_U^G(M). \quad (2.2)$$

Compositions, partitions and tableaux. A *composition* $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ is a sequence of positive integers. The *size* of μ is $|\mu| = \mu_1 + \mu_2 + \dots + \mu_r$, the *length* of μ is $\ell(\mu) = r$ and

$$\mathcal{B}_\mu = \{\mu_1, \mu_1 + \mu_2, \dots, \mu_1 + \mu_2 + \dots + \mu_r\}. \quad (2.3)$$

If $|\mu| = n$, then μ is a *composition of n* and we write $\mu \models n$. View μ as a collection of boxes aligned to the left; for example, if

$$\mu = (2, 5, 3, 4) = \begin{array}{ccccccc} & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{array},$$

then $|\mu| = 14$, $\ell(\mu) = 4$ and $\mathcal{B}_\mu = \{2, 7, 10, 14\}$. Alternatively, \mathcal{B}_μ coincides with the numbers in the boxes at the end of the rows in the diagram

$$\begin{array}{ccccccc} 1 & 2 & & & & & \\ 3 & 4 & 5 & 6 & 7 & & \\ 8 & 9 & 10 & & & & \\ 11 & 12 & 13 & 14 & & & \end{array}.$$

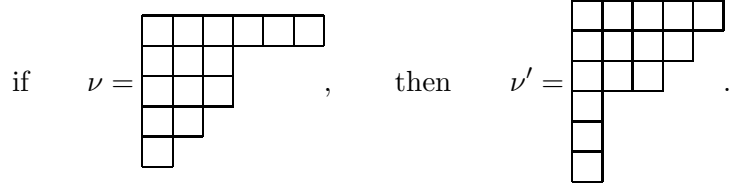
A *partition* $\nu = (\nu_1, \nu_2, \dots, \nu_r)$ is a composition where $\nu_1 \geq \nu_2 \geq \dots \geq \nu_r > 0$. If $|\nu| = n$, then ν is a *partition of n* and we write $\nu \vdash n$. Let

$$\mathcal{P} = \{\text{partitions}\} \quad \text{and} \quad \mathcal{P}_n = \{\nu \vdash n\}. \quad (2.4)$$

Suppose $\nu \in \mathcal{P}$. The *conjugate partition* $\nu' = (\nu'_1, \nu'_2, \dots, \nu'_\ell)$ is given by

$$\nu'_i = \text{Card}\{j : \nu_j \geq i\}.$$

In terms of diagrams, ν' is the collection of boxes obtained by flipping ν across its main diagonal. For example,



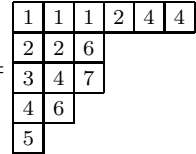
A *column strict tableau Q of shape ν* is a filling of the boxes of ν by positive integers such that

- (a) the entries strictly increase along columns,
- (b) the entries weakly increase along rows.

The *weight of Q* is the composition $\text{wt}(Q) = (\text{wt}(Q)_1, \text{wt}(Q)_2, \dots)$ given by

$$\text{wt}(Q)_i = \text{number of } i \text{ in } Q.$$

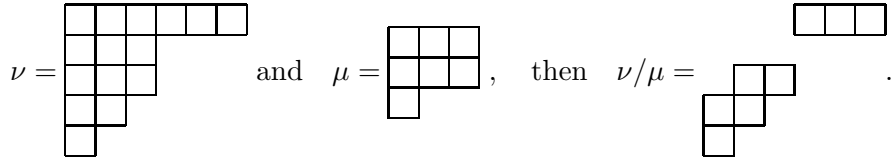
For example,

$$Q =$$
  $\text{ has } \text{wt}(Q) = (3, 3, 1, 4, 1, 2, 1).$

Suppose ν, μ are partitions. If $\nu_i \geq \mu_i$ for all $1 \leq i \leq \ell(\mu)$, then the *skew partition ν/μ* is given by

$$\nu/\mu = (\nu_1 - \mu_1, \nu_2 - \mu_2, \dots, \nu_{\ell(\nu)} - \mu_{\ell(\nu)}),$$

where $\mu_k = 0$ for all $k \geq \ell(\mu)$. In terms of boxes, represent ν/μ by removing μ from the upper left-hand corner of the diagram ν , so if



A *column strict tableau of shape ν/μ* is a filling of ν/μ satisfying (a) and (b) above.

Symmetric Functions. The symmetric group S_n acts on the infinite set of variables $\{x_1, x_2, \dots\}$ by permuting the indices $\leq n$ and fixing those $> n$. Let

$$\Lambda_{\mathbb{C}}(x) = \{f \in \mathbb{C}[[x_1, x_2, \dots]] : w(f) = f, \text{ permutations } w\}$$

be the ring of symmetric functions in the variables $\{x_1, x_2, \dots\}$. Let

$$e_r(x) = \sum_{1 \leq i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \quad \text{and} \quad p_s(x) = \sum_{1 \leq i} x_i^s, \quad r, s \in \mathbb{Z}_{\geq 0},$$

be the r th elementary symmetric function and the s th power sum symmetric function, respectively. For a partition $\nu = (\nu_1, \nu_2, \dots, \nu_\ell) \in \mathcal{P}$, let

$$e_\nu(x) = e_{\nu_1}(x) e_{\nu_2}(x) \cdots e_{\nu_\ell}(x), \quad p_\nu(x) = p_{\nu_1}(x) p_{\nu_2}(x) \cdots p_{\nu_\ell}(x)$$

and let

$$s_\nu(x) = \det(e_{\nu'_i - i + j}(x)) \quad (2.5)$$

be the Schur function corresponding to ν . The ring

$$\Lambda_{\mathbb{C}}(x) = \mathbb{C}\text{-span}\{e_\nu(x)\} = \mathbb{C}\text{-span}\{p_\nu(x)\} = \mathbb{C}\text{-span}\{s_\nu(x)\}, \quad (2.6)$$

and Pieri's rule says that if $\nu \in \mathcal{P}$, then

$$s_\nu(x) s_{(n)}(x) = \sum_{\substack{\text{sh}(P) = \gamma / \nu \\ \text{wt}(P) = (n)}} s_\gamma(x). \quad [\text{Ma, I.5.16}] \quad (2.7)$$

For each $t \in \mathbb{C}$ and partition ν , let $P_\nu(x; t)$ denote the Hall-Littlewood symmetric function [Ma, III.2]. Since a precise definition is not necessary for this paper, it suffices to remark that

$$P_\nu(x; 0) = s_\nu(x), \quad P_{(1^n)}(x; t) = e_n(x)$$

and for each $t \in \mathbb{C}$

$$\Lambda_{\mathbb{C}}(x) = \mathbb{C}\text{-span}\{P_\nu(x; t)\}.$$

(For additional details, see [Ma, Chapter I] on symmetric functions and [Ma, Chapter III] on Hall-Littlewood functions).

Remark. It is usually sufficient to let $\{x_1, x_2, \dots, x_K\}$ be a finite variable set (for K much bigger than $|G_n|$); think of symmetric functions as polynomials rather than formal power series by setting $x_j = 0$ for $j > K$ in the definitions above. While Theorem 5.2 requires the infinite definition, I urge the reader to think in terms of the finite version everywhere else.

RSK correspondence. The classical RSK correspondence provides a combinatorial proof of the identity

$$\prod_{i,j > 0} \frac{1}{1 - x_i y_j} = \sum_{\nu \vdash n, n \geq 0} s_\nu(x) s_\nu(y) \quad [\text{Kn}]$$

by constructing a bijection between the matrices $b \in M_\ell(\mathbb{Z}_{\geq 0})$ and the set of pairs $(P(b), Q(b))$ of column strict tableaux with the same shape. The bijection is as follows.

If P is a column strict tableau and $j \in \mathbb{Z}_{>0}$, let $P \leftarrow j$ be the column strict tableau given by the following algorithm

- (a) Insert j into the first column of P_k by displacing the smallest number $\geq j$. If all numbers are $< j$, then place j at the bottom of the first column.
- (b) Iterate this insertion by inserting the displaced entry into the next column.
- (c) Stop when the insertion does not displace an entry.

A *two-line array* $\begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$ is a two-rowed array with $i_1 \leq i_2 \leq \cdots \leq i_n$ and $j_k \geq j_{k+1}$ if $i_k = i_{k+1}$. If $b \in M_\ell(\mathbb{Z}_{\geq 0})$, then let \vec{b} be the two-line array with b_{ij} pairs $\binom{i}{j}$.

For $b \in M_\ell(\mathbb{Z}_{\geq 0})$, suppose

$$\vec{b} = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}.$$

Then the pair $(P(b), Q(b))$ is the final pair in the sequence

$$(\emptyset, \emptyset) = (P_0, Q_0), (P_1, Q_1), (P_2, Q_2), \dots, (P_n, Q_n) = (P(b), Q(b)),$$

where (P_k, Q_k) is a pair of column strict tableaux with the same shape given by

$$P_k = P_{k-1} \leftarrow j_k \quad \text{and} \quad Q_k \text{ is defined by } \text{sh}(Q_k) = \text{sh}(P_k) \text{ with } i_k \text{ in the new box } \text{sh}(Q_k)/\text{sh}(Q_{k-1}).$$

For example,

$$b = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{corresponds to} \quad \vec{b} = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 \\ 2 & 1 & 3 & 3 & 2 \end{pmatrix}$$

and provides the sequence

$$(\emptyset, \emptyset), (\boxed{2}, \boxed{1}), (\boxed{1 \ 2}, \boxed{1 \ 1}), \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 3 & 3 & 2 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 & 2 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 3 \\ \hline 2 & 3 & 3 & 2 \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 2 & 2 \\ \hline \end{array} \right)$$

so that

$$(P(b), Q(b)) = \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & 2 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 & 2 \\ \hline \end{array} \right).$$

The general linear group. Let $G = \text{GL}_n(\mathbb{F}_q)$, where \mathbb{F}_q is the finite field with q elements. Let

$$U = \left\{ \begin{pmatrix} 1 & * & \cdots & * \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\} \subseteq G$$

be the subgroup of unipotent, upper-triangular matrices. For $1 \leq i \neq j \leq n$, let $x_{ij}(t)$ be the matrix with ones on the diagonal, t in the (i, j) th position and zeroes elsewhere. Then

$$U = \langle x_{ij}(t) : 1 \leq i < j \leq n \rangle.$$

The group G has a double-coset decomposition given by

$$G = \bigsqcup_{v \in N} UvU, \quad \text{where } N = \left\{ \begin{array}{l} n \times n \text{ matrices with entries from } \\ \mathbb{F}_q \text{ and exactly one nonzero en-} \\ \text{try in each row and column} \end{array} \right\}. \quad (2.8)$$

If $T \subseteq N$ is the subgroup of diagonal matrices and $W \subseteq N$ is the the subgroup of permutation matrices, then $N = WT$ and $TU = \mathbf{N}_G(U)$ is the normalizer of U in G . If necessary, specify the size of the group by a subscript such as G_n , U_n , W_n , etc. Let $w_{(k)} \in W_k$ be the $k \times k$ matrix

$$(w_{(k)})_{ij} = \delta_{j, n-i+1}. \quad \text{For example, } w_{(3)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (2.9)$$

For $\mu = (\mu_1, \mu_2, \dots, \mu_\ell) \models n$,

$$P_\mu = \left\{ \begin{pmatrix} \boxed{g_1} & & * \\ & \boxed{g_2} & \\ & & \ddots \\ 0 & & & \boxed{g_\ell} \end{pmatrix} : g_i \in G_{\mu_i} = \text{GL}_{\mu_i}(\mathbb{F}_q) \right\} \quad (2.10)$$

is a *parabolic subgroup* of G . The *Levi subgroup* and the *unipotent radical* of P_μ are

$$L_\mu = \left\{ \begin{pmatrix} \boxed{g_1} & & 0 \\ & \boxed{g_2} & \\ & & \ddots \\ 0 & & & \boxed{g_\ell} \end{pmatrix} : g_i \in G_{\mu_i} \right\} \text{ and } U_\mu = \left\{ \begin{pmatrix} \boxed{Id_{\mu_1}} & & * \\ & \boxed{Id_{\mu_2}} & \\ & & \ddots \\ 0 & & & \boxed{Id_{\mu_\ell}} \end{pmatrix} \right\}, \quad (2.11)$$

respectively, where Id_k is the $k \times k$ identity matrix. Note that $P_\mu = L_\mu U_\mu$ and $P_\mu = \mathbf{N}_G(U_\mu)$.

3 An indexing for the standard basis of \mathcal{H}_μ

Let $G = \text{GL}_n(\mathbb{F}_q)$. Fix a nontrivial character $\psi : \mathbb{F}_q^+ \rightarrow \mathbb{C}^*$ of the additive group of \mathbb{F}_q . Let $\mu \models n$ and \mathcal{B}_μ be as in (2.3). Since $x_{ij}(t) \in [U, U]$ for all $j > i + 1$, the map $\psi_\mu : U \rightarrow \mathbb{C}^*$, defined by

$$\psi_\mu(x_{ij}(t)) = \begin{cases} \psi(t), & \text{if } j = i + 1, i \notin \mathcal{B}_\mu, \\ 1, & \text{otherwise,} \end{cases} \quad (3.1)$$

is a linear character of U . Let

$$\mathcal{H}_\mu = \text{End}_G(\text{Ind}_U^G(\psi_\mu)) \cong e_\mu \mathbb{C} G e_\mu, \quad \text{where } e_\mu = \frac{1}{|U|} \sum_{u \in U} \psi_\mu(u^{-1})u. \quad (3.2)$$

The classical examples of unipotent Hecke algebras are the Yokonuma algebra $\mathcal{H}_{(1^n)}$ [Yo2] and the Gelfand-Graev Hecke algebra $\mathcal{H}_{(n)}$ [St]. A fundamental result is

Theorem 3.1 ([GG],[Yo1],[St]). *For all $n > 0$, $\mathcal{H}_{(n)}$ is commutative.*

This theorem will follow from Theorem 5.3.

An analysis of (2.1) implies that $\{T_v = e_\mu v e_\mu : v \in N_\mu\}$ is a basis for \mathcal{H}_μ , where

$$N_\mu = \{v \in N : u, vuv^{-1} \in U \text{ implies } \psi_\mu(u) = \psi_\mu(vuv^{-1})\}. \quad [\text{CR}, (11.30)] \quad (3.3)$$

Suppose $a \in M_\ell(\mathbb{F}_q[X])$ is an $\ell \times \ell$ matrix with polynomial entries. Let $d(a_{ij})$ be the degree of the polynomial a_{ij} . Define the *degree row sums* and the *degree column sums* of a to be the compositions

$$d^\rightarrow(a) = (d^\rightarrow(a)_1, d^\rightarrow(a)_2, \dots, d^\rightarrow(a)_\ell) \quad \text{and} \quad d^\downarrow(a) = (d^\downarrow(a)_1, d^\downarrow(a)_2, \dots, d^\downarrow(a)_\ell),$$

where

$$d^\rightarrow(a)_i = \sum_{j=1}^{\ell} d(a_{ij}) \quad \text{and} \quad d^\downarrow(a)_j = \sum_{i=1}^{\ell} d(a_{ij}).$$

Let

$$M_\mu = \{a \in M_{\ell(\mu)}(\mathbb{F}_q[X]) : d^\rightarrow(a) = d^\downarrow(a) = \mu, a_{ij} \text{ monic}, a_{ij}(0) \neq 0\}. \quad (3.4)$$

$$\begin{pmatrix} X+1 & 1 & 1 & X+2 \\ X+3 & X^3+2X+3 & 1 & X+2 \\ 1 & X^2+4X+2 & X+2 & 1 \\ 1 & 1 & X^2+3X+1 & X^2+2 \end{pmatrix} \begin{matrix} (1+0+0+1=2) \\ (1+3+0+1=5) \\ (0+2+1+0=3) \\ (0+0+2+2=4) \end{matrix} \in M_{(2,5,3,4)}.$$
$$v_{(f)} = w_{(n)}(a_0 w_{(i_1)} \oplus a_1 w_{(i_2 - i_1)} \oplus \cdots \oplus a_r w_{(n - i_r)}) \in N, \quad (3.5)$$
$$v_{(a+bX^3+cX^4+X^6)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boxed{0} & \boxed{0} & \boxed{a} & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{0} & \boxed{c} \\ 0 & 0 & 0 & 0 & \boxed{c} & \boxed{0} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & b & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \end{pmatrix}.$$
[illegible]
$$\begin{array}{ccc} M_\mu & \longrightarrow & N_\mu \\ a & \mapsto & v_a, \end{array}$$

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Remarks. When $\mu = (n)$ this theorem says that the map $(f) \mapsto v_{(f)}$ of (3.5) is a bijection between $M_{(n)}$ and $N_{(n)}$.

Proof. Using the remark following the theorem, it is straightforward to reconstruct a from v_a . Therefore the map is invertible, and it suffices to show

- (a) the map is well-defined ($v_a \in N_\mu$),
- (b) the map is surjective.

To show (a) and (b), we investigate the matrices N_μ . Suppose $v \in N_\mu$. Let

$$\begin{aligned} v_i &= \text{the nonzero entry in the } i\text{th column of } v, \\ v(i) &= \text{the row number of the nonzero entry in the } i\text{th column of } v, \end{aligned}$$

so that $\{v_1, v_2, \dots, v_n\}$ are the nonzero entries of v and $(v(1), v(2), \dots, v(n))$ is the permutation determined by setting all the nonzero entries of v to 1. By (3.1),

$$\psi_\mu(x_{ij}(t)) = \begin{cases} \psi(t), & \text{if } j = i + 1 \text{ and } i \notin \mathcal{B}_\mu, \\ 1, & \text{otherwise.} \end{cases} \quad (\text{A})$$

Recall that $v \in N_\mu$ if and only if $u, vuv^{-1} \in U$ implies $\psi_\mu(u) = \psi_\mu(vuv^{-1})$. That is, $v \in N_\mu$ if and only if for all $1 \leq i < j \leq n$ such that $v(i) < v(j)$,

$$\begin{aligned} \psi_\mu(x_{ij}(t)) &= \psi_\mu(vx_{ij}(t)v^{-1}) \\ &= \psi_\mu(x_{v(i)v(j)}(v_itv_j^{-1})) \\ &= \begin{cases} \psi(v_itv_j^{-1}), & \text{if } v(j) = v(i) + 1 \text{ and } v(i) \notin \mathcal{B}_\mu, \\ 1, & \text{otherwise.} \end{cases} \end{aligned} \quad (\text{B})$$

Compare (A) and (B) to obtain that $v \in N_\mu$ if and only if for all $1 \leq i < j \leq n$ such that $v(i) < v(j)$,

- (i) If $i \notin \mathcal{B}_\mu$ and $v(i) \in \mathcal{B}_\mu$, then $j \neq i + 1$,
- (ii) If $i \in \mathcal{B}_\mu$ and $v(i) \notin \mathcal{B}_\mu$, then $v(j) \neq v(i) + 1$,
- (iii) If $i, v(i) \notin \mathcal{B}_\mu$, then $j = i + 1$ if and only if $v(j) = v(i) + 1$,
- (iii)' If $i, v(i) \notin \mathcal{B}_\mu$ and $v(j) = v(i) + 1$, then $v_i = v_{i+1}$.

We can visualize the implications of the conditions (i)–(iii)' in the following way. Partition the rows and columns of $v \in N_\mu$ by μ . For example, $\mu = (2, 3, 1)$ partitions v according to

$$\left(\begin{array}{cc|ccc|c} * & * & * & * & * & * \\ * & * & * & * & * & * \\ \hline * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ \hline * & * & * & * & * & * \end{array} \right).$$

Suppose the nonzero entry of v in column i is above a horizontal line but not next to a vertical line. Then condition (i) implies that $v(i+1) < v(i)$, so

$$\begin{array}{cccccc} & & 0 & * & & \\ & & 0 & * & & \\ & & 0 & * & & \\ \hline & 0 & 0 & a & 0 & 0 \end{array} \quad \begin{array}{l} \text{where } a \text{ is the nonzero entry and} \\ * \text{ indicates possible locations for} \\ \text{nonzero entries in the next column.} \end{array} \quad (\text{I})$$

Similarly, condition (ii) implies

$$\begin{array}{cccc|c} & & & & 0 \\ & & & & 0 \\ 0 & 0 & 0 & & a \\ * & * & * & & 0 \\ & & & & 0 \end{array} \quad (\text{II})$$

and conditions (iii) and (iii)' imply

$$\begin{array}{cccccc} & & 0 & * & & 0 & 0 \\ & & 0 & * & & 0 & 0 \\ & & 0 & * & & 0 & 0 & 0 & a & 0 & 0 & . \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \quad \text{or} \quad \begin{array}{cccccc} & & 0 & 0 & 0 & a & 0 & 0 & . \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \quad (\text{III})$$

In the case $\mu = (n)$ condition (III) implies that every $v \in N_{(n)}$ is of the form

$$\begin{pmatrix} 0 & & & \boxed{a_r Id_{i_r}} \\ & & \ddots & \\ & & \boxed{a_2 Id_{i_2}} & \\ \boxed{a_1 Id_{i_1}} & & & 0 \end{pmatrix} = w_{(n)}(a_1 w_{(i_1)} \oplus a_2 w_{(i_2)} \oplus \cdots \oplus a_r w_{(i_r)}),$$

where $(i_1, i_2, \dots, i_r) \models n$, $a_i \in \mathbb{F}_q^*$, and $w_{(k)} \in W_k$ is as in (2.9). In fact, this observation proves that the map $(f) \mapsto v_{(f)}$ is a bijection between $M_{(n)}$ and $N_{(n)}$ (mentioned in the remark).

Note that since the matrices v_a satisfy (I), (II) and (III), $v_a \in N_\mu$, proving (a). On the other hand, (I), (II) and (III) imply that each $v \in N_\mu$ must be of the form $v = v_a$ for some $a \in M_\mu$, proving (b). \square

4 An RSK-insertion via the representation theory of \mathcal{H}_μ

Let \mathcal{S} be a set. An \mathcal{S} -partition $\lambda = (\lambda^{(s_1)}, \lambda^{(s_2)}, \dots)$ is a sequence of partitions indexed by the elements of \mathcal{S} . Let

$$\mathcal{P}^\mathcal{S} = \{\mathcal{S}\text{-partitions}\}. \quad (4.1)$$

The following discussion defines two sets Θ and Φ , so that Θ -partitions index the irreducible characters of G and Φ -partitions index the conjugacy classes of G .

Let $L_n = \text{Hom}(\mathbb{F}_{q^n}^*, \mathbb{C}^*)$ be the character group of $\mathbb{F}_{q^n}^*$. If $\gamma \in L_m$, then let

$$\begin{aligned} \gamma_{(r)} : \mathbb{F}_{q^{mr}}^* &\longrightarrow \mathbb{C}^* \\ x &\longmapsto \gamma(x^{1+q^r+q^{2r}+\cdots+q^{m(r-1)}}) \end{aligned}$$

Thus if $n = mr$, then we may view $L_m \subseteq L_n$ by identifying $\gamma \in L_m$ with $\gamma_{(r)} \in L_n$. Define

$$L = \bigcup_{n \geq 0} L_n.$$

The *Frobenius maps* are

$$\begin{array}{ccc} F : \bar{\mathbb{F}}_q & \rightarrow & \bar{\mathbb{F}}_q \\ x & \mapsto & x^q \end{array} \quad \text{and} \quad \begin{array}{ccc} F : L & \rightarrow & L \\ \gamma & \mapsto & \gamma^q \end{array},$$

where $\bar{\mathbb{F}}_q$ is the algebraic closure of \mathbb{F}_q .

The map

$$\begin{array}{ccc} \{\text{F-orbits of } \bar{\mathbb{F}}_q^*\} & \longrightarrow & \{f \in \mathbb{F}_q[t] : f \text{ is monic, irreducible, and } f(0) \neq 0\} \\ \{x, x^q, x^{q^2}, \dots, x^{q^{k-1}}\} & \mapsto & f_x = \prod_{i=1}^{k-1} (t - x^{q^i}), \quad \text{where } x^{q^k} = x \in \bar{\mathbb{F}}_q^* \end{array}$$

is a bijection such that the size of the F -orbit of x equals the degree $d(f_x)$ of f_x . Let

$$\Phi = \left\{ f \in \mathbb{C}[t] : \begin{array}{l} f \text{ is monic, irre-} \\ \text{ducible and } f(0) \neq 0 \end{array} \right\} \quad \text{and} \quad \Theta = \{F\text{-orbits in } L\}. \quad (4.2)$$

If η is a Φ -partition and λ is a Θ -partition, then let

$$|\eta| = \sum_{f \in \Phi} d(f) |\eta^{(f)}| \quad \text{and} \quad |\lambda| = \sum_{\varphi \in \Theta} |\varphi| |\lambda^{(\varphi)}|$$

be the *size* of η and λ , respectively. Let the sets \mathcal{P}^Φ and \mathcal{P}^Θ be as in (4.1) and let

$$\mathcal{P}_n^\Phi = \{\eta \in \mathcal{P}^\Phi : |\eta| = n\} \quad \text{and} \quad \mathcal{P}_n^\Theta = \{\lambda \in \mathcal{P}^\Theta : |\lambda| = n\}. \quad (4.3)$$

Theorem 4.1 (Green [Gr]). *Let $G_n = \text{GL}_n(\mathbb{F}_q)$.*

- (a) \mathcal{P}_n^Φ indexes the conjugacy classes K^η of G_n ,
- (b) \mathcal{P}_n^Θ indexes the irreducible G_n -modules G_n^λ .

Suppose $\lambda \in \mathcal{P}^\Theta$. A *column strict tableau* $P = (P^{(\varphi_1)}, P^{(\varphi_2)}, \dots)$ of shape λ is a column strict filling of λ by positive integers. That is, $P^{(\varphi)}$ is a column strict tableau of shape $\lambda^{(\varphi)}$. Write $\text{sh}(P) = \lambda$. The *weight* of P is the composition $\text{wt}(P) = (\text{wt}(P)_1, \text{wt}(P)_2, \dots)$ given by

$$\text{wt}(P)_i = \sum_{\varphi \in \Theta} |\varphi| \left(\begin{array}{c} \text{number of} \\ i \text{ in } P^{(\varphi)} \end{array} \right).$$

If $\lambda \in \mathcal{P}^\Theta$ and μ is a composition, then let

$$\hat{\mathcal{H}}_\mu^\lambda = \{\text{column strict tableaux } P : \text{sh}(P) = \lambda, \text{wt}(P) = \mu\} \quad (4.4)$$

and

$$\hat{\mathcal{H}}_\mu = \{\lambda \in \mathcal{P}^\Theta : \hat{\mathcal{H}}_\mu^\lambda \text{ is not empty}\}. \quad (4.5)$$

The following theorem is a consequence of (2.2) and a theorem proved by Zelevinsky [Ze] (see Theorem 5.1). A proof of Zelevinsky's theorem is in Section 5.

Theorem 4.2. *The set $\hat{\mathcal{H}}_\mu$ indexes the irreducible \mathcal{H}_μ -modules \mathcal{H}_μ^λ and*

$$\dim(\mathcal{H}_\mu^\lambda) = |\hat{\mathcal{H}}_\mu^\lambda|.$$

The $(\mathcal{H}_\mu, \mathcal{H}_\mu)$ -bimodule decomposition

$$\mathcal{H}_\mu \cong \bigoplus_{\lambda \in \hat{\mathcal{H}}_\mu} \mathcal{H}_\mu^\lambda \otimes \mathcal{H}_\mu^\lambda \quad \text{implies} \quad |N_\mu| = \dim(\mathcal{H}_\mu) = \bigoplus_{\lambda \in \hat{\mathcal{H}}_\mu} \dim(\mathcal{H}_\mu^\lambda)^2 = \sum_{\lambda \in \hat{\mathcal{H}}_\mu} |\hat{\mathcal{H}}_\mu^\lambda|^2.$$

Theorem 4.3, below, gives a combinatorial proof of this identity.

Encode each matrix $a \in M_\mu$ as a Φ -sequence

$$(a^{(f_1)}, a^{(f_2)}, \dots), \quad f_i \in \Phi,$$

where $a^{(f)} \in M_{\ell(\mu)}(\mathbb{Z}_{\geq 0})$ is given by

$$a_{ij}^{(f)} = \text{highest power of } f \text{ dividing } a_{ij}.$$

Note that this is an entry by entry “factorization” of a such that

$$a_{ij} = \prod_{f \in \Phi} f^{a_{ij}^{(f)}}.$$

Recall from page 5 the classical RSK correspondence

$$\begin{aligned} M_\ell(\mathbb{Z}_{\geq 0}) &\longrightarrow \left\{ \begin{array}{l} \text{Pairs } (P, Q) \text{ of column strict} \\ \text{tableaux of the same shape} \end{array} \right\} \\ b &\mapsto (P(b), Q(b)). \end{aligned}$$

Theorem 4.3. *For $a \in M_\mu$, let $P(a)$ and $Q(a)$ be the Φ -column strict tableaux given by*

$$P(a) = (P(a^{(f_1)}), P(a^{(f_2)}), \dots) \quad \text{and} \quad Q(a) = (Q(a^{(f_1)}), Q(a^{(f_2)}), \dots) \quad \text{for } f_i \in \Phi.$$

Then the map

$$\begin{aligned} N_\mu &\longrightarrow M_\mu \longrightarrow \left\{ \begin{array}{l} \text{Pairs } (P, Q) \text{ of } \Phi\text{-column} \\ \text{strict tableaux of the same} \\ \text{shape and weight } \mu \end{array} \right\} \\ v &\mapsto a_v \mapsto (P(a_v), Q(a_v)), \end{aligned}$$

is a bijection, where the first map is the inverse of the bijection of Theorem 3.2.

By the construction above, the map is well-defined and since all the steps are invertible, the map is a bijection.

For example, suppose $\mu = (7, 5, 3, 2)$ and $f, g, h \in \Phi$ are such that $d(f) = 1$, $d(g) = 2$, and $d(h) = 3$. Then

$$a_v = \begin{pmatrix} g & f^2h & 1 & 1 \\ h & 1 & g & 1 \\ 1 & 1 & f & f^2 \\ g & 1 & 1 & 1 \end{pmatrix} \in M_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}}$$

corresponds to the sequence

$$(a_v^{(f_1)}, a_v^{(f_2)}, \dots) = \left(\begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{(f)}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^{(g)}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{(h)} \right)$$

and

$$(P(a_v), Q(a_v)) = \left(\begin{array}{|c|c|c|} \hline 2 & 2 & 4 \\ \hline 3 & 4 & \\ \hline \end{array}^{(f)}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array}^{(g)}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline & \\ \hline \end{array}^{(h)} \right) \left(\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 3 & 3 & \\ \hline \end{array}^{(f)}, \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline \end{array}^{(g)}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline & \\ \hline \end{array}^{(h)} \right).$$

5 Zelevinsky's decomposition of $\text{Ind}_U^G(\psi_\mu)$

This section proves the theorem

Theorem 5.1 (Zelevinsky [Ze]). *Let U be the subgroup of unipotent upper-triangular matrices of $G = \text{GL}_n(\mathbb{F}_q)$, $\mu \models n$ and ψ_μ be as in (3.1).*

$$\text{Ind}_U^G(\psi_\mu) = \bigoplus_{\lambda \in \hat{\mathcal{H}}_\mu} (G^\lambda)^{\oplus |\hat{\mathcal{H}}_\mu^\lambda|}.$$

Theorem 4.2 follows from this theorem and double-centralizer theory (2.2). The following will

- (i) establish the necessary connection between symmetric functions and the representation theory of G ,
- (ii) prove Theorem 5.1 for the case when $\ell(\mu) = 1$,
- (iii) generalize to arbitrary μ .

The proof below uses the ideas of Zelevinsky's proof, but explicitly uses symmetric functions to prove the results. Specifically, the following discussion through the proof of Theorem 5.3 corresponds to [Ze, Sections 9-11] and Theorem 5.1 corresponds to [Ze, Theorem 12.1].

Preliminaries to the proof. Suppose $\lambda \in \mathcal{P}^\Theta$ and $\eta \in \mathcal{P}^\Phi$ (see (4.3)). Let χ^λ be the irreducible character corresponding to the irreducible G -module G^λ and let κ^η be the characteristic function corresponding to the conjugacy class K^η (see Theorem 4.1), given by

$$\kappa^\eta(g) = \begin{cases} 1, & \text{if } g \in K^\eta, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } g \in G_{|\eta|}.$$

Define

$$R = \mathbb{C}\text{-span}\{\chi^\lambda : \lambda \in \mathcal{P}^\Theta\} = \mathbb{C}\text{-span}\{\kappa^\eta : \eta \in \mathcal{P}^\Phi\}.$$

The space R has an inner product defined by

$$\langle \chi^\lambda, \chi^\nu \rangle = \delta_{\lambda\nu},$$

and multiplication

$$\chi^\lambda \circ \chi^\nu = \text{Ind}_{L(r,s)}^{G_{r+s}}(\chi^\lambda \otimes \chi^\nu) = \text{Ind}_{P(r,s)}^{G_{r+s}} \left(\text{Inf}_{L(r,s)}^{P(r,s)}(\chi^\lambda \otimes \chi^\nu) \right), \quad \text{for } \lambda \in \mathcal{P}_r^\Theta, \nu \in \mathcal{P}_s^\Theta, \quad (5.1)$$

where if $p \in P_\mu = L_\mu U_\mu$ decomposes as $p = lu$ for $u \in U_\mu$, $l \in L_\mu$, then

$$\text{Inf}_{L_\mu}^{P_\mu}(\chi^{\gamma_1} \otimes \cdots \otimes \chi^{\gamma_\ell})(p) = \chi^{\gamma_1}(l_1) \cdots \chi^{\gamma_\ell}(l_\ell), \quad \text{for } l = (l_1 \oplus \cdots \oplus l_\ell), l_i \in G_{\mu_i}, \gamma_i \in \mathcal{P}_{\mu_i}^\Theta.$$

For each $\varphi \in \Theta$, let $\{Y_1^{(\varphi)}, Y_2^{(\varphi)}, \dots\}$ be an infinite set of variables, and let

$$\Lambda_{\mathbb{C}} = \bigotimes_{\varphi \in \Theta} \Lambda_{\mathbb{C}}(Y^{(\varphi)}),$$

where $\Lambda_{\mathbb{C}}(Y^{(\varphi)})$ is the ring of symmetric functions in $\{Y_1^{(\varphi)}, Y_2^{(\varphi)}, \dots\}$ (see page 4). For each $f \in \Phi$, define an additional set of variables $\{X_1^{(f)}, X_2^{(f)}, \dots\}$ such that the symmetric functions in the Y variables are related to the symmetric functions in the X variables by the transform

$$p_k(Y^{(\varphi)}) = (-1)^{k|\varphi|-1} \sum_{x \in \mathbb{F}_{q^{k|\varphi|}}^*} \xi(x) p_{\frac{k|\varphi|}{d(f_x)}}(X^{(f_x)}), \quad (5.2)$$

where $\xi \in \varphi$, $f_x \in \Phi$ is the irreducible polynomial that has x as a root, and $p_{\frac{a}{b}}(X^{(f)}) = 0$ if $\frac{a}{b} \notin \mathbb{Z}_{\geq 0}$. Then

$$\Lambda_{\mathbb{C}} = \bigotimes_{f \in \Phi} \Lambda_{\mathbb{C}}(X^{(f)}).$$

For $\nu \in \mathcal{P}$, let $s_\nu(Y^{(\varphi)})$ be the Schur function and $P_\nu(X^{(f)}; t)$ be the Hall-Littlewood symmetric function (see page 4). Define

$$s_\lambda = \prod_{\varphi \in \Theta} s_{\lambda^{(\varphi)}}(Y^{(\varphi)}) \quad \text{and} \quad P_\eta = q^{-n(\eta)} \prod_{f \in \Phi} P_{\eta^{(f)}}(X^{(f)}; q^{-d(f)}), \quad (5.3)$$

where $n(\eta) = \sum_{f \in \Phi} d(f)n(\eta^{(f)})$ and $n(\mu) = \sum_{i=1}^\ell (i-1)\mu_i$, for μ a composition. The ring

$$\Lambda_{\mathbb{C}} = \mathbb{C}\text{-span}\{s_\lambda : \lambda \in \mathcal{P}^\Theta\} = \mathbb{C}\text{-span}\{P_\eta : \eta \in \mathcal{P}^\Phi\}$$

has an inner product given by

$$\langle s_\lambda, s_\nu \rangle = \delta_{\lambda\nu}.$$

Theorem 5.2 (Green [Gr], Macdonald [Ma]). *The linear map*

$$\begin{aligned} \text{ch} : R &\longrightarrow \Lambda_{\mathbb{C}} \\ \chi^\lambda &\mapsto s_\lambda, & \text{for } \lambda \in \mathcal{P}^\Theta \\ \kappa^\eta &\mapsto P_\eta, & \text{for } \eta \in \mathcal{P}^\Phi, \end{aligned}$$

is an algebra isomorphism that preserves the inner product.

The unipotent conjugacy classes K^η satisfy $\eta^{(f)} = \emptyset$ unless $f = t-1$. Let

$$\mathcal{U} = \mathbb{C}\text{-span}\{\kappa^\eta : \eta^{(f)} = \emptyset, \text{ unless } f = t-1\} \subseteq R$$

be the subalgebra of unipotent class functions. Note that by (5.3) and Theorem 5.2

$$\text{ch}(\mathcal{U}) = \Lambda_{\mathbb{C}}(X^{(t-1)}).$$

Consider the projection $\pi : R \rightarrow \mathcal{U}$ which is an algebra homomorphism given by

$$(\pi\chi^\lambda)(g) = \begin{cases} \chi^\lambda(g), & \text{if } g \in G \text{ is unipotent,} \\ 0, & \text{otherwise,} \end{cases} \quad \lambda \in \mathcal{P}^\Theta.$$

Then $\tilde{\pi} = \pi \circ \text{ch}^{-1} : \Lambda_{\mathbb{C}} \rightarrow \mathbb{C}$ is given by

$$\begin{aligned} \tilde{\pi}(p_k(Y^{(\varphi)})) &= \tilde{\pi} \left((-1)^{k|\varphi|-1} \sum_{x \in \mathbb{F}_{q^{k|\varphi|}}^*} \xi(x) p_{\frac{k|\varphi|}{d(fx)}}(X^{(fx)}) \right) \quad (\text{by (5.2)}) \\ &= (-1)^{k|\varphi|-1} \xi(1) p_{\frac{k|\varphi|}{1}}(X^{(t-1)}) + 0 \\ &= (-1)^{k|\varphi|-1} p_{k|\varphi|}(X^{(t-1)}). \end{aligned} \quad (5.4)$$

The decomposition of $\text{Ind}_U^G(\psi_{(n)})$. The representation $\text{Ind}_U^G(\psi_{(n)})$ is the Gelfand-Graev module, and with (2.2), Theorem 5.3 proves that $\mathcal{H}_{(n)}$ is commutative.

For $\lambda \in \mathcal{P}^\Theta$, let

$$\text{ht}(\lambda) = \max\{\ell(\lambda^{(\varphi)}) : \varphi \in \Theta\}.$$

Theorem 5.3.

$$\text{ch}(\text{Ind}_U^G(\psi_{(n)})) = \sum_{\lambda \in \mathcal{P}_n^\Theta, \text{ht}(\lambda)=1} s_\lambda.$$

Proof. Let

$$\begin{aligned} \Psi : R &\longrightarrow \mathbb{C} \\ \chi^\lambda &\mapsto \langle \chi^\lambda, \text{Ind}_U^G(\psi_{(n)}) \rangle \end{aligned} \quad \text{and} \quad \tilde{\Psi} : \Lambda_{\mathbb{C}} \xrightarrow{\text{ch}^{-1}} R \xrightarrow{\Psi} \mathbb{C}. \quad (5.5)$$

For any group H , let 1_H be the trivial character of H , $e_H = \frac{1}{|H|} \sum_{h \in H} h$, and $\langle \chi, \gamma \rangle_H = \frac{1}{|H|} \sum_{h \in H} \chi(h) \gamma(h^{-1})$, for all class functions γ and χ of H .

The proof is in six steps.

- (a) $\tilde{\Psi}(e_k(Y^{(1)})) = \delta_{k1}$, where 1 is the trivial character of \mathbb{F}_q^* ,
- (b) $\Psi(\chi^\lambda) = \dim(e_{(n)} G^\lambda)$ for $\lambda \in \mathcal{P}^\Theta$,
- (c) $\tilde{\Psi}(fg) = \tilde{\Psi}(f)\tilde{\Psi}(g)$ for all $f, g \in \Lambda_{\mathbb{C}}(Y^{(1)})$, where 1 is the trivial character of \mathbb{F}_q^* ,
- (d) $\Psi \circ \pi = \Psi$,
- (e) $\tilde{\Psi}(f(Y^{(\varphi)})) = \tilde{\Psi}(f(Y^{(1)}))$ for all $f \in \Lambda_{\mathbb{C}}(Y^{(\varphi)})$,
- (f) $\tilde{\Psi}(s_\lambda) = \delta_{\text{ht}(\lambda)1}$.

(a) An argument similar to the argument in [Ma, pgs. 285-286] shows that

$$\text{ch}^{-1}(e_k(Y^{(1)})) = 1_{G_k}$$

(see [HR, Theorem 4.9 (a)] for details). Therefore, by Frobenius reciprocity and the orthogonality of characters,

$$\tilde{\Psi}(e_k(Y^{(1)})) = \langle 1_{G_k}, \text{Ind}_U^G(\psi_{(n)}) \rangle = \langle 1_{U_k}, \psi_{(n)} \rangle_{U_k} = \delta_{k1}.$$

(b) Since there exists an idempotent e such that $G^\lambda \cong \mathbb{C}Ge$ and $\text{Ind}_U^G(\psi_{(n)}) \cong \mathbb{C}Ge_{(n)}$, the map

$$\begin{aligned} e_{(n)} \mathbb{C}Ge &\longrightarrow \text{Hom}_G(G^\lambda, \mathbb{C}Ge_{(n)}) \\ e_{(n)} ge &\mapsto \begin{aligned} \gamma_g : \mathbb{C}Ge &\rightarrow \mathbb{C}Ge_{(n)} \\ xe &\mapsto xege_{(n)} \end{aligned} \end{aligned}$$

is a vector space isomorphism (using an argument similar to the proof of [CR, (3.18)]). Thus,

$$\Psi(\chi^\lambda) = \langle \chi^\lambda, \text{Ind}_U^G(\psi_{(n)}) \rangle = \dim(\text{Hom}_G(G^\lambda, \text{Ind}_U^G(\psi_{(n)}))) = \dim(e_{(n)}\mathbb{C}Ge) = \dim(e_{(n)}G^\lambda).$$

(c) By (a), $\tilde{\Psi}(e_r(Y^{(1)}))\tilde{\Psi}(e_s(Y^{(1)})) = \delta_{r1}\delta_{s1}$. Since $\Lambda_{\mathbb{C}}(Y^{(1)}) = \mathbb{C}[e_1(Y^{(1)}), e_2(Y^{(1)}), \dots]$, it therefore suffices to show that

$$\tilde{\Psi}(e_r(Y^{(1)})e_s(Y^{(1)})) = \delta_{r1}\delta_{s1}.$$

Suppose $r + s = n$ and let $P = P_{(r,s)}$. Then

$$\tilde{\Psi}(e_r(Y^{(1)})e_s(Y^{(1)})) = \Psi(\text{Ind}_P^{G_n}(1_P)) = \dim(e_{(n)}\mathbb{C}Ge_P).$$

Since $T \subseteq P$, $e_P = e_{(1^n)}e_P$, $G = \bigsqcup_{v \in N} UvU$, and $N = WT$,

$$e_{(n)}\mathbb{C}Ge_P = e_{(n)}\mathbb{C}Ge_{(1^n)}e_P = \mathbb{C}\text{-span}\{e_{(n)}we_{(1^n)}e_P : w \in W\}.$$

If there exists $1 \leq i \leq n$ such that $w(i) = w(i) + 1$, then

$$e_{(n)}we_{(1^n)} = e_{(n)}wx_{i,i+1}(t)e_{(1^n)} = e_{(n)}x_{w(i),w(i)+1}(t)we_{(1^n)} = \psi(t)e_{(n)}we_{(1^n)}.$$

Therefore, $e_{(n)}we_{(1^n)} = 0$ unless $w = w_{(n)}$. If $r > 1$ or $s > 1$, then there exists $1 \leq i \leq n$ such that $x_{i+1,i}(t) \in P_{(r,s)}$, so

$$e_{(n)}w_{(n)}e_P = e_{(n)}w_{(n)}x_{i+1,i}(t)e_P = e_{(n)}x_{n-i,n-i+1}(t)w_{(n)}e_P = \psi(t)e_{(n)}w_{(n)}e_P = 0.$$

In particular,

$$\dim(e_{(n)}\mathbb{C}Ge_P) = 0.$$

If $r = s = 1$, then $P_{(1,1)}$ is upper-triangular, so

$$e_{(2)}w_{(2)}e_P \neq 0$$

and $\dim(e_{(2)}\mathbb{C}Ge_P) = 1$, giving $\tilde{\Psi}(e_r(Y^{(1)})e_s(Y^{(1)})) = \delta_{r1}\delta_{s1}$.

(d) By Frobenius reciprocity,

$$\langle \chi^\lambda, \text{Ind}_{U_n}^{G_n}(\psi_{(n)}) \rangle = \langle \text{Res}_{U_n}^{G_n}(\chi^\lambda), \psi_{(n)} \rangle_{U_n} = \langle \text{Res}_{U_n}^{G_n}(\pi(\chi^\lambda)), \psi_{(n)} \rangle_{U_n} = \langle \pi(\chi^\lambda), \text{Ind}_{U_n}^{G_n}(\psi_{(n)}) \rangle,$$

so $\Psi = \Psi \circ \pi$.

(e) Induct on n , using (c) and the identity

$$(-1)^{n-1}p_n(Y^{(1)}) = ne_n(Y^{(1)}) - \sum_{r=1}^{n-1} (-1)^{r-1}p_r(Y^{(1)})e_{n-r}(Y^{(1)}), \quad [\text{Ma, I.2.11}']$$

to obtain $\tilde{\Psi}(p_n(Y^{(1)})) = 1$. Note that

$$\begin{aligned} \tilde{\Psi}(p_n(Y^{(\varphi)})) &= \tilde{\Psi}(\pi(p_n(Y^{(\varphi)}))) = \tilde{\Psi}((-1)^{|\varphi|n-1}p_{|\varphi|n}(X^{(t-1)})) = \tilde{\Psi}(\pi(p_{|\varphi|k}(Y^{(1)}))) \\ &= \tilde{\Psi}(p_{|\varphi|k}(Y^{(1)})) = 1 = \tilde{\Psi}(p_k(Y^{(1)})). \end{aligned}$$

Since $\tilde{\Psi}$ is multiplicative on $\Lambda_{\mathbb{C}}(Y^{(1)})$,

$$\tilde{\Psi}(p_\nu(Y^{(\varphi)})) = 1 = \tilde{\Psi}(p_\nu(Y^{(1)})), \quad \text{for all partitions } \nu.$$

In particular, since $\tilde{\Psi}$ is linear and $\Lambda_{\mathbb{C}}(Y^{(\varphi)}) = \mathbb{C}\text{-span}\{p_{\nu}(Y^{(\varphi)})\}$,

$$\tilde{\Psi}(f(Y^{(\varphi)})) = \tilde{\Psi}(f(Y^{(1)})), \quad \text{for all } f \in \Lambda_{\mathbb{C}}(Y^{(\varphi)}).$$

Note that (e) also implies that $\tilde{\Psi}$ is multiplicative on all of $\Lambda_{\mathbb{C}}$.

(f) Note that

$$\tilde{\Psi}(s_{\lambda}) = \tilde{\Psi}\left(\prod_{\varphi \in \Theta} s_{\lambda(\varphi)}(Y^{(\varphi)})\right) = \tilde{\Psi}\left(\prod_{\varphi \in \Theta} s_{\lambda(\varphi)}(Y^{(1)})\right) = \prod_{\varphi \in \Theta} \tilde{\Psi}(s_{\lambda(\varphi)}(Y^{(1)})),$$

where the last two equalities follow from (e) and (c), respectively. By definition $s_{\nu}(Y^{(1)}) = \det(e_{\nu'_i - i + j}(Y^{(1)}))$, so

$$\tilde{\Psi}(s_{\nu}(Y^{(1)})) = \begin{cases} 1, & \text{if } \ell(\nu) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

implies

$$\text{ch}(\text{Ind}_U^G(\psi_{(n)})) = \sum_{\lambda \in \mathcal{P}_n^{\Theta}} \tilde{\Psi}(s_{\lambda}) s_{\lambda} = \sum_{\lambda \in \mathcal{P}_n^{\Theta}, \text{ht}(\lambda)=1} s_{\lambda}. \quad \square$$

Decomposition of $\text{Ind}_U^G(\psi_{\mu})$. Suppose $\lambda, \nu \in \mathcal{P}^{\Theta}$. A *column strict tableau* P of shape λ and *weight* ν is a column strict filling of λ such that for each $\varphi \in \Theta$,

$$\text{sh}(P^{(\varphi)}) = \lambda^{(\varphi)} \quad \text{and} \quad \text{wt}(P^{(\varphi)}) = \nu^{(\varphi)}.$$

We can now prove the theorem stated at the beginning of this section:

Theorem 5.1 ([Ze]) Let U be the subgroup of unipotent upper-triangular matrices of $G = \text{GL}_n(\mathbb{F}_q)$, $\mu \models n$ and ψ_{μ} be as in (3.1). Then

$$\text{Ind}_U^G(\psi_{\mu}) = \bigoplus_{\lambda \in \mathcal{H}_{\mu}} (G^{\lambda})^{\oplus |\mathcal{H}_{\mu}^{\lambda}|}.$$

Proof. Note that

$$\text{Ind}_U^{P_{\mu}}(\psi_{\mu}) \cong \mathbb{C}P_{\mu}e_{\mu} = \mathbb{C}P_{\mu}e_{[\mu]}e'_{[\mu]},$$

where

$$e_{[\mu]} = \frac{1}{|U \cap L_{\mu}|} \sum_{u \in U \cap L_{\mu}} \psi_{\mu}(u^{-1})u \quad \text{and} \quad e'_{[\mu]} = \frac{1}{|U_{\mu}|} \sum_{u \in U_{\mu}} u. \quad (5.6)$$

Thus,

$$\begin{aligned} \text{Ind}_U^{P_{\mu}}(\psi_{\mu}) &\cong \text{Inf}_{L_{\mu}}^{P_{\mu}}\left(\text{Ind}_{U \cap L_{\mu}}^{L_{\mu}}(\psi_{\mu})\right) \\ &\cong \text{Inf}_{L_{\mu}}^{P_{\mu}}\left(\text{Ind}_{U_{\mu_1}}^{G_{\mu_1}}(\psi_{(\mu_1)}) \otimes \text{Ind}_{U_{\mu_2}}^{G_{\mu_2}}(\psi_{(\mu_2)}) \otimes \cdots \otimes \text{Ind}_{U_{\mu_{\ell}}}^{G_{\mu_{\ell}}}(\psi_{(\mu_{\ell})})\right) \end{aligned}$$

In particular, by the definition of multiplication in R (5.1),

$$\Gamma_{\mu} = \text{ch}(\text{Ind}_U^G(\psi_{\mu})) = \Gamma_{\mu_1} \Gamma_{\mu_2} \cdots \Gamma_{\mu_{\ell}}, \quad \text{where } \Gamma_{\mu_i} = \sum_{\lambda \in \mathcal{P}_{\mu_i}^{\Theta}, \text{ht}(\lambda)=1} s_{\lambda}.$$

Pieri's rule (2.7) implies that for $\lambda \in \mathcal{P}_r^\Theta$, $\nu \in \mathcal{P}_s^\Theta$ and $\text{ht}(\nu) = 1$,

$$s_\lambda s_\nu = \sum_{\gamma \in \mathcal{P}_{r+s}^\Theta, |\hat{\mathcal{H}}_\nu^{\gamma/\lambda}| \neq 0} s_\gamma, \quad \text{so} \quad \Gamma_\mu = \sum_{\lambda \in \mathcal{P}^\Theta} K_{\lambda\mu} s_\lambda,$$

where

$$\begin{aligned} K_{\lambda\mu} &= \text{Card}\{\emptyset = \gamma_0 \subset \gamma_1 \subset \gamma_2 \subset \cdots \subset \gamma_\ell = \lambda : |\hat{\mathcal{H}}_{(\mu_{i+1})}^{\gamma_{i+1}/\gamma_i}| = 1\} \\ &= \text{Card}\{\text{column strict tableaux of shape } \lambda \text{ and weight } \mu\} = |\hat{\mathcal{H}}_\mu^\lambda|. \end{aligned}$$

By Green's Theorem (Theorem 5.2), ch is an isomorphism, so

$$\text{Ind}_U^G(\psi_\mu) = \text{ch}^{-1}(\Gamma_\mu) = \sum_{\lambda \in \hat{\mathcal{H}}_\mu} |\hat{\mathcal{H}}_\mu^\lambda| \text{ch}^{-1}(s_\lambda) = \sum_{\lambda \in \hat{\mathcal{H}}_\mu} (G^\lambda)^{\oplus |\hat{\mathcal{H}}_\mu^\lambda|}. \quad \square$$

6 A weight space decomposition of \mathcal{H}_μ -modules

Let $\mu = (\mu_1, \mu_2, \dots, \mu_\ell) \models n$ and let P_μ , L_μ and U_μ be as in (2.10) and (2.11). Recall that

$$e_\mu = \frac{1}{|U|} \sum_{u \in U} \psi_\mu(u^{-1})u.$$

Theorem 6.1. *For $a \in M_\mu$, let $T_a = e_\mu v_a e_\mu$ with v_a as in (3.6). Then the map*

$$\begin{aligned} \mathcal{H}_{(\mu_1)} \otimes \mathcal{H}_{(\mu_2)} \otimes \cdots \otimes \mathcal{H}_{(\mu_\ell)} &\longrightarrow \mathcal{H}_\mu \\ T_{(f_1)} \otimes T_{(f_2)} \otimes \cdots \otimes T_{(f_\ell)} &\mapsto T_{(f_1) \oplus (f_2) \oplus \cdots \oplus (f_\ell)}, \quad \text{for } (f_i) \in M_{(\mu_i)} \end{aligned}$$

is an injective algebra homomorphism with image $\mathcal{L}_\mu = e_\mu P_\mu e_\mu = e_\mu L_\mu e_\mu$.

Proof. Note that

$$T_{(f_1)} \otimes T_{(f_2)} \otimes \cdots \otimes T_{(f_\ell)} = \frac{1}{|U \cap L_\mu|^2} \sum_{x_i, y_i \in U_{\mu_i}} \left(\prod_{i=1}^{\ell} \psi_\mu(x_i^{-1} y_i^{-1}) \right) x_1 v_{(f_1)} y_1 \otimes x_2 v_{(f_2)} y_2 \otimes \cdots \otimes x_\ell v_{(f_\ell)} y_\ell.$$

Since $U = (L_\mu \cap U)(U_\mu)$, $L_\mu \cap U \cong U_{\mu_1} \times U_{\mu_2} \times \cdots \times U_{\mu_\ell}$, and ψ_μ is trivial on U_μ ,

$$\begin{aligned} T_{(f_1) \oplus (f_2) \oplus \cdots \oplus (f_\ell)} &= \frac{1}{|U|^2} \sum_{x, y \in U} \psi_\mu(x^{-1} y^{-1}) x(v_{(f_1)} \oplus v_{(f_2)} \oplus \cdots \oplus v_{(f_\ell)}) y \\ &= \frac{1}{|U \cap L_\mu|^2} \sum_{x_i, y_i \in U_{\mu_i}} \psi_\mu(x_1^{-1} y_1^{-1} \oplus \cdots \oplus x_\ell^{-1} y_\ell^{-1}) e'_{[\mu]} x_1 v_{(f_1)} y_1 \oplus \cdots \oplus x_\ell v_{(f_\ell)} y_\ell e'_{[\mu]}, \end{aligned}$$

where $e'_{[\mu]}$ is as in (5.6). Since $L_\mu \subseteq \mathbf{N}_G(U_\mu)$, the idempotent $e'_{[\mu]}$ commutes with $x_1 v_{(f_1)} y_1 \oplus \cdots \oplus x_\ell v_{(f_\ell)} y_\ell$ and

$$T_{(f_1) \oplus (f_2) \oplus \cdots \oplus (f_\ell)} = \frac{e'_{[\mu]}}{|L \cap U|^2} \sum_{x_i, y_i \in U_{\mu_i}} \left(\prod_{i=1}^{\ell} \psi_\mu(x_i^{-1} y_i^{-1}) \right) x_1 v_{(f_1)} y_1 \oplus \cdots \oplus x_\ell v_{(f_\ell)} y_\ell.$$

Consequently, the map multiplies by $e'_{[\mu]}$ and changes \otimes to \oplus , so it is an algebra homomorphism. Since the map sends basis elements to basis elements, it is also injective. \square

Let \mathcal{L}_μ be as in Theorem 6.1. By Theorem 3.1 each $\mathcal{H}_{(\mu_i)}$ is commutative, so \mathcal{L}_μ is commutative and all the irreducible \mathcal{L}_μ -modules \mathcal{L}_μ^γ are one-dimensional. Theorem 4.2 implies that

$$\hat{\mathcal{H}}_{(\mu_i)} = \{\Theta\text{-partitions } \lambda : |\lambda| = \mu_i, \text{ht}(\lambda) = 1\}.$$

indexes the irreducible $\mathcal{H}_{(\mu_i)}$ -modules. Therefore, the set

$$\hat{\mathcal{L}}_\mu = \hat{\mathcal{H}}_{(\mu_1)} \times \hat{\mathcal{H}}_{(\mu_2)} \times \cdots \times \hat{\mathcal{H}}_{(\mu_\ell)} = \{\gamma = (\gamma_1, \gamma_2, \dots, \gamma_\ell) : \gamma_i \in \hat{\mathcal{H}}_{(\mu_i)}\} \quad (6.1)$$

indexes the irreducible \mathcal{L}_μ -modules. Identify $\gamma \in \hat{\mathcal{L}}_\mu$ with the map $\gamma : \mathcal{L}_\mu \rightarrow \mathbb{C}$ such that

$$yv = \gamma(y)v, \quad \text{for all } y \in \mathcal{L}_\mu, v \in \mathcal{L}_\mu^\gamma.$$

For $\gamma \in \hat{\mathcal{L}}_\mu$, define the γ -weight space V_γ of an \mathcal{H}_μ -module V to be

$$V_\gamma = \{v \in V : yv = \gamma(y)v, \text{ for all } y \in \mathcal{L}_\mu\}.$$

Then

$$V \cong \bigoplus_{\gamma \in \hat{\mathcal{L}}_\mu} V_\gamma.$$

Let $\lambda \in \mathcal{P}^\Theta$ and $\gamma \in \hat{\mathcal{L}}_\mu$. A *column strict tableau* P of shape λ and weight γ is column strict filling of λ such that for each $\varphi \in \Theta$,

$$\text{sh}(P^{(\varphi)}) = \lambda^{(\varphi)} \quad \text{and} \quad \text{wt}(P^{(\varphi)}) = (|\gamma_1^{(\varphi)}|, |\gamma_2^{(\varphi)}|, \dots, |\gamma_\ell^{(\varphi)}|),$$

where $|\gamma_i^{(\varphi)}|$ is the number of boxes in the partition $\gamma_i^{(\varphi)}$ (which has length 1).

Theorem 6.2. *Let \mathcal{H}_μ^λ be an irreducible \mathcal{H}_μ -module and $\gamma \in \hat{\mathcal{L}}_\mu$. Then*

$$\dim(\mathcal{H}_\mu^\lambda)_\gamma = \text{Card}\{\text{column strict tableaux of shape } \lambda \text{ and weight } \gamma\} = |\hat{\mathcal{H}}_\gamma^\lambda|.$$

Proof. By double-centralizer theory and Frobenius reciprocity,

$$\dim((\mathcal{H}_\mu^\lambda)_\gamma) = \langle \text{Res}_{\mathcal{L}_\mu}^{\mathcal{H}_\mu}(\mathcal{H}_\mu^\lambda), \mathcal{L}_\mu^\gamma \rangle = \langle \text{Res}_{P_\mu}^G(G^\lambda), P_\mu^\gamma \rangle = \langle G^\lambda, \text{Ind}_{P_\mu}^G(P_\mu^\gamma) \rangle,$$

where $P_\mu^\gamma = \text{Inf}_{L_\mu}^{P_\mu}(L_\mu^\gamma)$. Therefore,

$$\dim((\mathcal{H}_\mu^\lambda)_\gamma) = c_\gamma^\lambda, \quad \text{where} \quad s_{\gamma_1} s_{\gamma_2} \cdots s_{\gamma_\ell} = \sum_{\lambda \in \mathcal{P}^\Theta} c_\gamma^\lambda s_\lambda.$$

Pieri's rule (2.7) implies $c_\gamma^\lambda = |\hat{\mathcal{H}}_\gamma^\lambda|$. □

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